BIOMAGNETIC INVERSE SOLUTION
USING COMBINED-NORM AND POINTWISE NORMALIZATION

Y. Terazono[1], A. Matani[1]

[1]Graduate School of Frontier Sciences, University of Tokyo, Japan

Abstract-**In this paper, we propose a method of solving the biomagnetic inverse problem consisting of two approaches, the first of which is pointwise normalization. In the conventional normalization technique, each variable is normalized individually. This variablewise normalization is appropriate for scalar fields, but not for vector fields where one vector on a grid point is represented by several variables. Hence, pointwise operation is needed for vector fields. The second is a combination of norms. Use of the ℓ_2-norm as the cost function of an optimization problem is known to lead to spatially spread solutions, while the ℓ_1-norm leads to sparse solutions. To control sparseness of solutions, we propose to use an internal division of the ℓ_2-norm and pointwise normalized ℓ_1-norm. The optimization problem constructed as above can be recast as a second-order cone program (SOCP), a nonlinear convex problem. The problem can be solved using recently developed efficient interior-point methods. Computer simulations showed that the sparseness of estimators obtained with the proposed method reflects both the ratio of internal division and the sparseness of true sources. Regularization of normalization and relaxation of constraint conditions in the presence of noise are also presented.**

Keywords - **Inverse problem, MEG, norm, SOCP, normalization**

I. INTRODUCTION

Estimation of the source current distribution from the measured biomagnetic field is called a biomagnetic inverse problem and has received much attention for many years. One of these problems is the MEG (magnetoencephalography) inverse problem, which is to reconstruct the source current distribution from the magnetic field generated by the brain.

The problem is usually underdetermined such that the solution is not unique. One approach to determining the solution uniquely is to formulate an optimization problem. An example of cost functions of optimization problems is the ℓ_2-norm of the solution. The minimal ℓ_2-norm solution can be obtained analytically by using the pseudoinverse matrix, and is known to be spread spatially. Another example is the ℓ_1-norm. In this case, the problem is solved by using linear programming. The minimal ℓ_1-norm solution is known to be sparse.

Normalization of variables was proposed for the use of the ℓ_1-norm [1]. The normalization technique is operated variablewise on the source vector and is appropriate for scalar fields. There is, however, room for taking into account that current distributions are vector fields. The problem formulation and these conventional methods are described in Section II.

In Section III, we propose a new normalization method, pointwise normalization. This can be successfully applied to vector fields. In addition, regularization of normalization is presented. We also propose a cost function that is an internal

division of the ℓ_2-norm and the pointwise normalized ℓ_1-norm. Computer simulation results of the proposed method are shown in Section IV. In Section V, a consideration of noise is included.

II. PRELIMINARIES

A. System Equation

The relationship between a biomagnetic field and a current density distribution is expressed as the linear equation

$$m = Lq, \qquad (1)$$

where $m \in \mathbf{R}^M$, $q \in \mathbf{R}^{3N}$, and $L \in \mathbf{R}^{M \times 3N}$ are the measured magnetic field data vector (M: number of sensors), source current distribution of discretized points in the source region ($3N$: number of points including orthogonal coordinate system), and leadfield matrix (or transfer matrix), respectively. Since the number of unknown variables is greater than the number of equations, $M < N$, the inverse problem will be underdetermined. Here, let us assume $\text{rank}[L] = M$. In (1), q is the unknown source to be estimated.

B. Inverse Solution

Representing a solution as a map g and the resulting estimator as a vector $\hat{q}$, we obtain

$$\hat{q} = g(m) = g(Lq). \qquad (2)$$

For example, any linear inverse solution g can be expressed by the matrix $G \in \mathbf{R}^{3N \times M}$ as

$$\hat{q} = Gm, \qquad (3)$$

and thus (2) can be written explicitly.

However g is not necessarily written as a function with an explicit form. An example is the case where the estimator is given as the solution of an optimization problem.

$$g(m) = \arg\min_{q} \left\{ S(q) \,\middle|\, m = Lq \right\} \qquad (4)$$

$S(q)$ is the cost function to be minimized under constraint (1).

C. Performance Criterion

One way to design a solution g is to introduce a performance criterion on g. Here we consider the square sum of estimation errors for fundamental vectors,

$$E(g, L, \{e_i\}) = \sum_{i=1}^{3N} \| g(Le_i) - e_i \|_2^2. \qquad (5)$$

$E > 0$ holds for any linear g if the problem is underde-

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Abstract

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termined. G (a linear g) minimizing (5) is the pseudoinverse matrix of L :

$$G = L^+ = L^T \left(L L^T \right)^{-1} . \qquad (6)$$

The important point to observe is that the same estimator as (6) is obtained by the following optimization problem:

$$g(m) = \arg\min_{q} \left\{ \|q\|_2 \,\middle|\, m = Lq \right\} . \qquad (7)$$

Although a linear g satisfying $E = 0$ yields the true solution, it cannot occur due to the singularity of L in (1). Generally speaking, a nonlinear g can be focused more point sources through the minimization of E, whereas the estimators for their superposition are usually not the superposition of the estimators for each point source.

D. Normalized ℓ₁-norm

Matsuura and Okabe [1] proposed in an optimization problem employing a variablewise normalized ℓ_1-norm as the cost function:

$$g_1(m) = D^{-1} \arg\min_{\tilde{q}} \left\{ S_1(q) \,\middle|\, m = \tilde{L}\tilde{q} \right\} , \qquad (8)$$

where $\tilde{L} = LD^{-1}$, $\tilde{q} = Dq$, and D is a diagonal matrix whose diagonal terms are $\|l_1\|_2, \ldots, \|l_{3N}\|_2$. l_i is the i th column vector of L. Thus we have $\tilde{l}_i = l_i \|l_i\|^{-1}$ and $\tilde{q}_i = \|l_i\| q_i$, i.e., the variablewise normalization. The cost function $S_1(q)$ in (8) is

$$S_1(q) = \|\tilde{q}\|_1 = \sum_{i=1}^{3N} \|l_i q_i\|_2 . \qquad (9)$$

The optimization problem can be formulated as a linear program. Solutions are known to be sparse with their number of nonzero variables being equal to or smaller than M if L is nondegenerate. Particularly, for point sources, g_1 in (8) successfully achieves $E = 0$.

Note that the variablewise normalization presented above has the intended affect only when source q is a scalar field. If not, normalization that allows vector fields is required. This case will be further investigated in the next section.

III. PROPOSED METHOD
A. Pointwise Normalization

Denoting the transfer matrix of point i by $L_i \in \mathbf{R}^{M \times 3}$, (1) can be rewritten as

$$m = \begin{bmatrix} L_1 & \cdots & L_N \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} , \qquad (10)$$

where $q_i \in \mathbf{R}^3$ is the source vector on the i th grid point.

In the case of vector fields, more than one variables related to the same grid point altogether represent the vector on that point. Those variables in general produce m s not orthogonal to each other. This nonorthogonality causes the variablewise normalization to be unsuitable for vector fields.

For vector fields, we propose a pointwise normalization instead of the variablewise normalization. That is, instead of (8), we use

$$g_{p1}(m) = H^{-1} \arg\min_{\tilde{q}} \left\{ S_{p1}(q) \,\middle|\, m = \tilde{L}\tilde{q} \right\} , \qquad (11)$$

where $\tilde{L} = LH^{-1}$ and $\tilde{q} = Hq$. H is a block diagonal matrix given as follows:

$$L_i^T L_i = V_i E_i V_i^T$$
$$H_i = E_i^{1/2} V_i^T \qquad (12)$$
$$H = \begin{bmatrix} H_1 & & \\ & \ddots & \\ & & H_N \end{bmatrix} ,$$

where eigenvalue decomposition of $L_i^T L_i$ is utilized. Thus we have $\tilde{L}_i = L_i V_i E_i^{-1/2}$ and $\tilde{q}_i = E_i^{1/2} V_i^T q_i$, i.e., the pointwise normalization. Let

$$r_i = \|\tilde{q}_i\|_2$$
$$r = [r_1, \ldots, r_N]^T , \qquad (13)$$

and the cost function $S_{p1}(q)$

$$S_{p1}(q) = \|r\|_1 = \sum_{i=1}^{N} \|\tilde{q}_i\|_2 = \sum_{i=1}^{N} \|L_i q_i\|_2 . \qquad (14)$$

This is the ℓ_1-norm of the vector r, which consists of the pointwise normalized norm of each point.

By solving (11) the following theorem holds.

Theorem Suppose we are solving (1), or equivalent, (10). Let L_i and L_j $(i \neq j)$ have no common subspace other than 0 (zero vector), i.e.,

$$\text{rank}[L_i \, L_j] = \text{rank}[L_i] + \text{rank}[L_j] = 6, \text{ if } i \neq j , \qquad (15)$$

and each pointwise transfer matrix be of full rank, i.e.,

$$\text{rank}[L_i] = 3 . \qquad (16)$$

Then, any point source is correctly estimated by solving (11). A point source with active point a is:

$$q_i \begin{cases} \neq 0, & i = a \\ = 0, & i \neq a \end{cases} . \qquad (17)$$

The proof is not included due to space limitation.

This result includes $E(g_{p1}, L, \{e_i\}) = 0$ and the fact that even if there are more than one nonzero variables in q, q is correctly solved when the nonzero variables are on the same discretized point. While the former is already shown in [1] under a stronger assumption than (15) and (16), the nondegeneracy assumption, the latter cannot be achieved by the variablewise normalization.

In norm minimizations, a point that produces smaller magnetic fields than other points tends to have a smaller estimated value. Normalization is intended to help avoiding such a dependence of the estimated values on measurement properties. Estimators, however, for source distributions other

than point sources could be perturbed if the matrix H^TH has a large condition number. We use a regularization technique to contend with this issue:

$$H_i' = (D_i + \alpha I)^{1/2} V_i^T \qquad (18)$$

$$S_{p1}(q,\alpha) = \sum_{i=1}^{N} \|H_i' q_i\|_2 , \qquad (19)$$

where α is the regularization parameter for normalization. Estimations are thought to be more stable for larger α, whereas normalization has no effect as $\alpha \to \infty$.

B. Combination of Norms

So far, we have seen that the linearity of solution g and performance $E(g,L,\{e_i\}) = 0$ are incompatible. Because g is desired to have both properties, a compromise between linearity and performance for point sources should be made.

One approach is to linearly combine an estimator by linear estimation (6), which is the same as (7), and one by estimation (11). The obtained estimator, however, has no additional information compared to the two original estimators and indeed, seems useless in practice.

In contrast, using internal division of the cost functions of (7) and (11) leads to an interesting estimator. The new cost function, the combined-norm, is

$$S_{p12}(q,\alpha,\beta) = \beta\|q\|_2 + (1-\beta)S_{p1}(\alpha,q), \qquad (20)$$

where β, $0 \le \beta \le 1$ is the ratio of the internal division. The estimators for different β show sparse to spread characteristics as β varies from 0 to 1.

The proposed cost function (20) is easily shown to be a norm. Thus the problem

$$g_{p12}(m,\alpha,\beta) = \arg\min_q \{S_{p12}(q,\alpha,\beta)|m = Lq\} \qquad (21)$$

is a convex optimization problem. Indeed, this problem (21) is readily reformulated as a second-order cone program (SOCP) [2]. Although this reformulation includes the relaxation of limiting conditions, equivalence of the optimization problem is preserved. For SOCPs, several efficient interior-point methods have been developed (see [3] for example.)
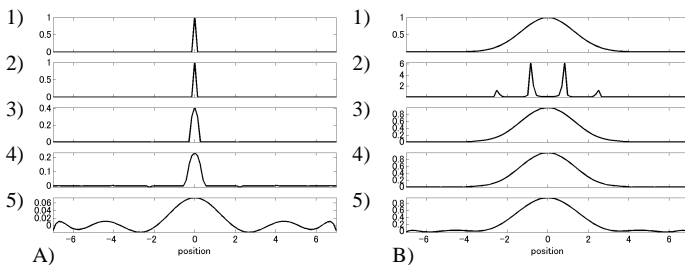

Fig. 1. The estimated results for scalar fields on a line, without normalization: A) a point source, B) a Gaussian spread source. 1) The true distribution, 2) the min ℓ_1-norm estimator, 3) min combined-norm with $\beta = 0.1$, 4) min combined-norm with $\beta = 0.3$, and 5) min ℓ_2-norm. It is shown that the spreads of the minimum combined-norm estimators correspond to that of the true distributions, while those of the minimum ℓ_1- and ℓ_2-norm estimators do not.

IV. RESULTS

A. Scalar Fields on a Line

Figure 1 shows estimators for a point source and a spread source, both on a line. The simulation system is the same as the one used in [4], i.e., a one-dimensional model.

In combined-norm cases, the spread of estimators reflects the internal division ratio and the spread of the true distribution, while the spreads of both the minimum ℓ_1-norm and ℓ_2-norm estimators do not.

B. a Vector Field on a Plane

Estimated results for a vector field on a plane having two active regions are illustrated in Fig. 2. The plane is assumed to be disk shaped with radius 8.06cm and placed at 4cm height in a spherical conductor. Magnetic fields are observed by 64 sensors, whose radiuses are from 10.4cm to 15.2cm. The number of grid points, N, is 688.

In the minimization of the ℓ_1-norm and combined-norm, the regularization of the pointwise normalization is performed such that the condition number after the operation $\lambda_{max}(H^TH + \alpha I)/\lambda_{min}(H^TH + \alpha I) = 10^4$, where $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are the maximum and the minimum eigenvalue of matrix A, respectively. Without the regularization, $\lambda_{max}(H^TH)/\lambda_{min}(H^TH) = 1.06 \times 10^7$.

For the distribution shown in Fig. 2, $\beta = 0.01$ seems appropriate for approximating the source. Note that the effect of the mixing parameter, β, will differ if the system transfer matrix and/or the number of variables, $3N$, differ.

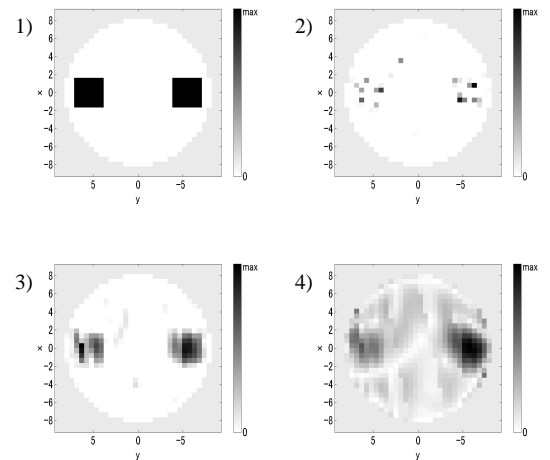## V. DISCUSSION AND CONCLUSIONS

Fig. 2. The estimated results for a vector field on a plane. The darkness of each dot corresponds to the norm of the vector on each point. 1) The true distribution, 2) the minimum pointwise normalized ℓ_1-norm estimator, 3) min combined-norm with $\beta = 0.01$, 4) min ℓ_2-norm. The normalization is regularized.

In practice, noise is inevitably mixed into measurements. Assuming noise has a normal distribution, relaxation of constraint conditions as follows may be a suitable approach:

$$S_{p12n}(\boldsymbol{q}, \alpha, \beta, \gamma) = \gamma S_{p12}(\boldsymbol{q}, \alpha, \beta) + (1 - \gamma)\|\boldsymbol{m} - \boldsymbol{Lq}\|_2 \quad (22)$$

$$g_{p12n}(\boldsymbol{m}, \alpha, \beta, \gamma) = \arg\min_{\boldsymbol{q}} \{S_{p12n}(\boldsymbol{q}, \alpha, \beta, \gamma)\}. \quad (23)$$

Problem (23) is still an SOCP. The second term on the right hand side of (22) may be modified by the noise covariance matrix.

In this paper, we proposed a pointwise normalization method, which can solve point sources correctly when used with the ℓ_1-norm, to handle vector fields appropriately. The regularization of the normalization is also introduced in order to obtain stable estimations. Then, a function derived by internal division of the ℓ_2-norm and pointwise normalized ℓ_1-norm is proposed as a cost function of the optimization problem. The problem is solved by reformulating it to an SOCP; the solution shows sparse or spread characteristics according to the ratio of internal division.

The next step in the research is to determine function parameters both in the case where one has some a priori knowledge about source activity and where nothing is assumed previously.

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